Collision group and renormalization of the Boltzmann collision integral

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On the basis of a recently discovered collision group [V. L. Saveliev, in *Rarefied Gas Dynamics: 22nd International Symposium*, edited by T. J. Bartel and M. Gallis, AIP Conf. Proc. No. 585 (AIP, Melville, NY, 2001), p. 101], the Boltzmann collision integral is exactly rewritten in two parts. The first part describes the scattering of particles with small angles. In this part the *infinity* due to the infinite cross sections is extracted from the Boltzmann collision integral. Moreover, the Boltzmann collision integral is represented as a divergence of the flow in velocity space. Owing to this, the role of collisions in the kinetic equation can be interpreted in terms of the nonlocal friction force that depends on the distribution function.

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I. INTRODUCTION

A. Parametrization of collisions by a rotation matrix and collision group

Applications of the group theory remain underutilized in kinetic theory although they deserve to be utilized [1,2]. In this paper, opportunities and advantages due to the parametrization of two-particle collisions by a matrix belonging to the group of rotations are shown. In order to construct the collision integral in the Boltzmann kinetic equation, a collision of two particles is usually determined by setting a direction $\boldsymbol{n}$ ($\boldsymbol{n}^2=1$) of the postcollisional relative velocity [3]. In that case, if a particle of mass m_1 and velocity $\boldsymbol{\nu}$ collides with a particle of mass m_2 and velocity $\boldsymbol{u}$, the velocities after collision are given by the following equations:

$$\boldsymbol{\nu}'=\frac{m\boldsymbol{\nu}+\boldsymbol{u}+|\boldsymbol{\nu}-\boldsymbol{u}|\boldsymbol{n}}{1+m},$$

$$\boldsymbol{u}'=\frac{m\boldsymbol{\nu}+\boldsymbol{u}-m|\boldsymbol{\nu}-\boldsymbol{u}|\boldsymbol{n}}{1+m}, \tag{1}$$

where m_1/m_2.

Also, there is another interesting parametrization [4] that naturally appears in the hard sphere collisions. This will be discussed in detail later. In accordance with the parametrization used in Eq. (1), the collision integral that describes the time evolution of the distribution function $f(\boldsymbol{\nu})$ of the first species due to collisions with the second species having the velocity distribution function $\psi(\boldsymbol{u})$ is given by the usual expression [5]

$$I(f,\psi)=\int v\sigma_\theta\left(v^2,\frac{\boldsymbol{v}\cdot\boldsymbol{n}}{v}\right)[f(\boldsymbol{\nu}')\psi(\boldsymbol{u}')-f(\boldsymbol{\nu})\psi(\boldsymbol{u})]d\Omega_n d\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{v}=\boldsymbol{\nu}-\boldsymbol{u}$ is the relative velocity and σ_θ is the differential collision cross section.

In the present paper, we propose to construct the collision integral using a parametrization of the scattering represented by a rotation matrix $\hat{R}\in O_3^+$, which is determined as follows by Euler's angles φ, θ, and ψ [6]:

$$\hat{R}(\varphi,\theta,\psi)=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}\cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\theta & -\cos\varphi\sin\psi-\sin\varphi\cos\psi\cos\theta & \sin\varphi\sin\theta\\ \sin\varphi\cos\psi+\cos\varphi\sin\psi\cos\theta & -\sin\varphi\sin\psi+\cos\varphi\cos\psi\cos\theta & -\cos\varphi\sin\theta\\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta\end{pmatrix}. \tag{3}$$

In this case, the transformation of the velocities due to a collision becomes a linear one, which is different from Eq. (1):

$$\boldsymbol{\nu}'=\frac{m\boldsymbol{\nu}+\boldsymbol{u}+\hat{R}(\boldsymbol{\nu}-\boldsymbol{u})}{1+m},$$

$$\boldsymbol{u}'=\frac{m\boldsymbol{\nu}+\boldsymbol{u}-m\hat{R}(\boldsymbol{\nu}-\boldsymbol{u})}{1+m}. \tag{4}$$

The velocities $\boldsymbol{\nu}'$ and $\boldsymbol{u}'$ are determined by a partitioned matrix (2×2 cells). The size of each cell is obviously (3×3):

$$\xi' = \hat{S}\xi, \quad \xi = \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{u} \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} \dfrac{m+\hat{R}}{1+m} & \dfrac{1-\hat{R}}{1+m} \\ \dfrac{m(1-\hat{R})}{1+m} & \dfrac{1+m\hat{R}}{1+m} \end{pmatrix}. \tag{5}$$

Here, ξ is a six-dimensional bivector consisting of the components $\boldsymbol{\nu}$ and $\boldsymbol{u}$. Let us briefly describe basic properties of these matrices that can be easily confirmed by their direct multiplication. The scattering matrices $\hat{S}(\hat{R})(\hat{R} \in O_3)$ constitute a group that is isomorphic to the group of orthogonal matrices O_3 (the group of rotations including improper rotations):

$$\hat{S}(\hat{R}_1)\cdot\hat{S}(\hat{R}_2) = \hat{S}(\hat{R}_1\cdot\hat{R}_2), \quad \hat{S}^{-1}(\hat{R}) = \hat{S}(\hat{R}^{-1}). \tag{6}$$

Here, it should be noted that parametrization [4] of a collision by the direction $\boldsymbol{n}$ normal to the plane of reflection of the relative velocity $(\boldsymbol{v}' = \boldsymbol{v} - 2\boldsymbol{nn}\cdot\boldsymbol{v})$ also provides a linear transformation of the particles' velocities due to collision. In this case, however, scattering matrices do not constitute a group, which is crucial for our further consideration. Due to energy conservation law for a collision, the matrix $\hat{S}(\hat{R})$ satisfies the following condition:

$$\tilde{\hat{S}}\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\hat{S} = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

where $\tilde{\hat{S}}$ is the transpose of $\hat{S}$. From Eq. (7) we have

$$|\det\hat{S}| = 1. \tag{8}$$

It provides the equality of velocity volumes:

$$d\boldsymbol{\nu}\, d\boldsymbol{u} = d\boldsymbol{\nu}'\, d\boldsymbol{u}'. \tag{9}$$

This relation is much simpler than $d\boldsymbol{\nu}\, d\boldsymbol{u}\, d\Omega' = d\boldsymbol{\nu}'\, d\boldsymbol{u}'\, d\Omega$ that appears in the case of the conventional [5] parametrization of a collision.

B. Collision integral

To rewrite the collision integral (2) for the case when a collision is parametrized by rotation matrix (3), integration over directions of the vector $\boldsymbol{n}$ should be replaced by integration on the invariant measure [6–8] over the group $O_3^+(\int d\Omega_n/4\pi \to \int d\hat{R}/8\pi^2)$:

$$d\hat{R} = d\psi\, d\varphi \sin\theta\, d\theta \quad (0 \leqslant \psi, \varphi \leqslant 2\pi,\ 0 \leqslant \theta \leqslant \pi),$$

$$d\hat{R} = d\hat{R}_0\hat{R} = d\hat{R}\hat{R}_0 = d\hat{R}^{-1}, \tag{10}$$

$$\int d\hat{R} = 8\pi^2.$$

The collision integral now takes the following form:

$$I(f,\psi) = I(F) = \int b(v,\mu)[f(\boldsymbol{\nu}')\psi(\boldsymbol{u}') - f(\nu)\psi(\boldsymbol{u})]\frac{d\hat{R}}{2\pi}d\boldsymbol{u}$$

$$= \int b(v,\mu)[F(\hat{S}(\hat{R})\xi) - F(\xi)]\frac{d\hat{R}}{2\pi}d\boldsymbol{u}, \tag{11}$$

where $b(v,\mu) = v\sigma_\theta(v,\mu)$ is a scattering indicatrix, $\mu(\hat{R},\boldsymbol{v}) = \boldsymbol{v}\cdot\hat{R}\boldsymbol{v}/v^2$ is the cosine of the scattering angle, and $F(\xi) = f\circ\psi(\xi) = f(\boldsymbol{\nu})\psi(\boldsymbol{u})$ is the two-particle velocity distribution function.

Here we demonstrate some simple advantages, which provide the form of the Boltzmann collision integral (11). By taking into consideration the following equation,

$$\delta(\boldsymbol{\nu}' - \boldsymbol{\nu}_0)\,\delta(\boldsymbol{u}' - \boldsymbol{u}_0) = \delta(\hat{S}\xi - \xi_0) = \frac{1}{|\det\hat{S}|}\,\delta(\xi - \hat{S}^{-1}\xi_0)$$

$$= \delta(\xi - \hat{S}(\hat{R}^{-1})\xi_0), \tag{12}$$

the measure invariance (10), and the property of the indicatrix ensuring a detailed equilibrium $b(\boldsymbol{v},\hat{R}) = b(\boldsymbol{v},\hat{R}^{-1})$, it can be easily seen that the expression for the kernel of the bilinear collision operator (11) can be rewritten in the following form:

$$I(\delta(\boldsymbol{\nu} - \boldsymbol{\nu}_0), \delta(\boldsymbol{u} - \boldsymbol{u}_0))$$

$$= I(\delta(\xi - \xi_0))$$

$$= \int d\boldsymbol{u}\frac{d\hat{R}}{2\pi}b(\boldsymbol{v},\hat{R})[\delta(\xi - \hat{S}(\hat{R}^{-1})\xi_0) - \delta(\xi - \xi_0)]$$

$$= \int \frac{d\hat{R}}{2\pi}b(\boldsymbol{v}_0,\hat{R})[\delta(\boldsymbol{\nu} - \boldsymbol{\nu}_0') - \delta(\boldsymbol{\nu} - \boldsymbol{\nu}_0)]. \tag{13}$$

A similar equation can also be obtained for the Maxwellian velocity clusters:

$$I(f_M(\boldsymbol{\nu} - \boldsymbol{\nu}_0), \psi_M(\boldsymbol{u} - \boldsymbol{u}_0))$$

$$= I(f_M\circ\psi_M(\xi - \xi_0))$$

$$= \int d\boldsymbol{u}\frac{d\hat{R}}{2\pi}b(\boldsymbol{v},\hat{R})[f_M\circ\psi_M(\xi - \hat{S}(\hat{R}^{-1})\xi_0)$$

$$- f_M\circ\psi_M(\xi - \xi_0)]$$

$$= \int \frac{d\hat{R}}{2\pi}[b_\tau(\boldsymbol{\nu},\boldsymbol{u}_0',\hat{R})f_M(\boldsymbol{\nu} - \boldsymbol{\nu}_0') - b_\tau(\boldsymbol{\nu},\boldsymbol{u}_0,\hat{R})f_M(\boldsymbol{\nu} - \boldsymbol{\nu}_0)], \tag{14}$$

where

$$b_\tau(\boldsymbol{\nu},\boldsymbol{u}_0,\hat{R}) = \int b(\boldsymbol{v},\hat{R})\psi_M(\boldsymbol{u} - \boldsymbol{u}_0)d\boldsymbol{u}, \tag{15}$$

$$f_M(\boldsymbol{\nu}) = \left(\frac{m_1}{2\pi kT}\right)^{3/2}\exp\left(-\frac{m_1\boldsymbol{\nu}^2}{2kT}\right),$$

$$\psi_M(\boldsymbol{u})=\left(\frac{m_2}{2\pi kT}\right)^{3/2}\exp\left(-\frac{m_2\boldsymbol{u}^2}{2kT}\right).$$

Here, Eqs. (7) and (8) were employed. At the zero temperature limit ($T\to 0$), from Eqs. (14) and (15), we have an asymptotic form similar to Eq. (13),

$$I(f_M(\boldsymbol{\nu}-\boldsymbol{\nu}_0)\psi_M(\boldsymbol{u}-\boldsymbol{u}_0))_{\tau\to 0}$$

$$=\int\frac{d\hat{R}}{2\pi}b(\boldsymbol{v}_0,\hat{R})[f_M(\boldsymbol{\nu}-\boldsymbol{\nu}_0')-f_M(\boldsymbol{\nu}-\boldsymbol{\nu}_0)]+O(T). \tag{16}$$

In the case of Maxwellian molecules, where the indicatrix b does not depend on the relative velocity (and consequently $b_T=b$), Eq. (16) is satisfied at any temperature and is a corollary of the statement of the invariance of the collision operator for a constant collision frequency with respect to the Gaussian transformation [9,2]:

$$e^{(kT/2m_1)/\nabla_\nu^2}I(f,\psi)=I(e^{(kT/2m_1)/\nabla_\nu^2}f,e^{(kT/2m_2)/\nabla_u^2}\psi), \tag{17}$$

where

$$e^{(kT/2m)/\nabla_\nu^2}f(\boldsymbol{\nu})=\left(\frac{m}{2\pi kT}\right)^{3/2}\int d\boldsymbol{\nu}'\,e^{-m(\boldsymbol{\nu}-\boldsymbol{\nu}')^2/2kT}f(\boldsymbol{\nu}'),$$

$$\nabla_\nu^2=\left(\frac{\partial}{\partial\boldsymbol{\nu}}\right)^2,\quad \nabla_u^2=\left(\frac{\partial}{\partial\boldsymbol{u}}\right)^2. \tag{18}$$

It should be noted that the invariance property of collision operator (17) for the case $m_1=m_2$ was utilized in a well-known paper [10] to obtain the exact solution of the Boltzmann equation.

The fact [Eq. (6)] that collision matrixes constitute a group gives us essentially new opportunities for investigating the Boltzmann equation. In a paper [1] of one of the present authors, a new method to directly construct a class of discrete velocity models for mixtures from the Boltzmann equation was proposed by replacing the integration of Eq. (11) over the full group of rotations by summation over the discrete subgroups of this group. In the following sections we make use of the group property (6) of the collision matrixes (5), and present a new exponential form of the collision operator and a method of renormalization of collision integral (11).

A long-standing problem in the kinetic theory is how to effectively describe, within the framework of the Boltzmann equation, the evolution of a system of particles interacting with long-range forces, especially with Coulomb forces. Many interesting methods (see [11,12] and references cited therein) have been developed for practical calculations of the velocity distribution function of Coulomb particles. It is expected that the exact renormalization of the Boltzmann collision integral proposed in the present paper will be an important step in solving these problems.

II. REPRESENTATION OF THE GROUP OF SCATTERING MATRICES IN THE HILBERT SPACE AND THE BOLTZMANN COLLISION INTEGRAL

There are many equivalent ways to parametrize rotation matrixes, each of which has its own advantage. We will here use a parametrization of proper rotations (reflections are not included) by the angle ($0\leqslant\phi\leqslant\pi$) of rotation and by the direction $\boldsymbol{n}$ of the axis of rotation, where $\boldsymbol{n}$ is the unit vector. Rotation of the angle ϕ around the axis $\boldsymbol{n}$ is given by the following formula [6,7]:

$$\hat{R}=e^{\phi\hat{n}}=(1-\cos\phi)\hat{n}^2+(\sin\phi)\hat{n}+1, \tag{19}$$

where the operator (matrix) $\hat{n}$ is defined by the following expression:

$$\hat{n}\equiv\boldsymbol{n}\times,\quad \hat{n}\boldsymbol{\nu}\equiv\boldsymbol{n}\times\boldsymbol{\nu}. \tag{20}$$

The linear span of the operators $\hat{n}$ constitutes the Lie algebra corresponding to the group of rotations with the following commutative relations:

$$[\hat{n}_1,\hat{n}_2]\equiv\hat{n}_1\hat{n}_2-\hat{n}_2\hat{n}_1=[\boldsymbol{n}_1\times\boldsymbol{n}_2]\times. \tag{21}$$

The infinitesimal rotations take the form, in accordance with the exponential representation of rotation matrix (19), of

$$\hat{R}=e^{\phi\hat{n}}\underset{\phi\to 0}{\simeq}1+\phi\hat{n}. \tag{22}$$

Substituting Eq. (22) into Eq. (5) for the scattering matrix, we obtain the following equation for the infinitesimal scattering matrix

$$\hat{S}=\begin{pmatrix}\dfrac{m+\hat{R}}{1+m} & \dfrac{1-\hat{R}}{1+m}\\[2mm] \dfrac{m(1-\hat{R})}{1+m} & \dfrac{1+m\hat{R}}{1+m}\end{pmatrix}\simeq 1+\phi\hat{c}\quad(\phi\to 0), \tag{23}$$

where the generator $\hat{c}$ for the scattering matrix $\hat{S}$ has the form

$$\hat{c}=\frac{1}{1+m}\begin{pmatrix}\hat{n} & -\hat{n}\\ -m\hat{n} & m\hat{n}\end{pmatrix}. \tag{24}$$

Making use of Eq. (24) for the generator $\hat{c}$, we can obtain an exponential representation for the scattering matrix

$$\hat{S}=e^{\phi\hat{c}}. \tag{25}$$

In accordance with the usual rules [6,7] of the theory of Lie groups, we can easily construct a representation of the group of scattering matrix $\hat{S}$ in the Hilbert space of functions on bivector variable ξ:

$$\hat{T}_sF(\xi)=F(\hat{S}^{-1}\xi). \tag{26}$$

To obtain the infinitesimal transformation $\hat{T}_s$ in the Hilbert space, we substitute Eq. (23) into Eq. (26),

$$\hat{T}_s F(\xi)=F(\hat{S}^{-1}\xi) \underset{\phi\to 0}{\simeq} F((1-\phi\hat{c})\xi)$$
$$=\left(1-\phi c_{i,k}\xi_k\frac{\partial}{\partial\xi_1}\right)F(\xi)$$
$$=\left(1-\phi\xi\tilde{\hat{c}}\frac{\partial}{\partial\xi}\right)F(\xi). \tag{27}$$

By taking into account that

$$c_{i,k}\xi_k\frac{\partial}{\partial\xi_1}=c_{i,k}\left(\frac{\partial}{\partial\xi_i}\xi_k-\delta_{i,k}\right)=\frac{\partial}{\partial\xi_i}c_{i,k}\xi_k, \tag{28}$$

it can easily be seen from Eq. (27) that the generator of the group of collision transformation $\hat{T}_s$ in the Hilbert space is

$$\hat{\sigma}=-\xi\tilde{\hat{c}}\frac{\partial}{\partial\xi}=-\frac{\partial}{\partial\xi}\hat{c}\,\xi. \tag{29}$$

The generators $\hat{\sigma}$ can be represented using $\boldsymbol{\nu}$ and $\boldsymbol{u}$:

$$\hat{\sigma}=-\frac{\partial}{\partial\xi}\hat{c}\,\xi=\frac{1}{1+m}\left(\frac{\partial}{\partial\boldsymbol{\nu}},\ \frac{\partial}{\partial\boldsymbol{u}}\right)\begin{pmatrix}-\hat{n} & \hat{n}\\ m\hat{n} & -m\hat{n}\end{pmatrix}\begin{pmatrix}\boldsymbol{\nu}\\ \boldsymbol{u}\end{pmatrix}$$
$$=\frac{1}{1+m}\left\{\boldsymbol{n}\cdot\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)\times(\boldsymbol{\nu}-\boldsymbol{u})\right\}. \tag{30}$$

It is seen from Eq. (30) that we can express the generator $\hat{\sigma}$ by the vector operator $\hat{\boldsymbol{\sigma}}$:

$$\hat{\sigma}=\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}, \tag{31}$$

where

$$\hat{\boldsymbol{\sigma}}=-\frac{1}{1+m}\boldsymbol{v}\times\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right). \tag{32}$$

The square of the vector operator $\hat{\boldsymbol{\sigma}}$ is as follows:

$$\hat{\boldsymbol{\sigma}}^2=\frac{1}{(1+m)^2}\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)(\boldsymbol{v}^2-\boldsymbol{v}\boldsymbol{v})\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)$$
$$=\frac{1}{(1+m)^2}\left(\frac{\partial}{\partial\nu_i}-m\frac{\partial}{\partial u_i}\right)(v^2\delta_{ik}-v_iv_k)\left(\frac{\partial}{\partial\nu_k}-m\frac{\partial}{\partial u_k}\right). \tag{33}$$

To better understand the algebraic properties of the operator $\hat{\boldsymbol{\sigma}}$, let us change variables. Introducing the relative velocity $\boldsymbol{v}=\boldsymbol{\nu}-\boldsymbol{u}$ and the center-of-mass velocity $\boldsymbol{w}=(m\boldsymbol{\nu}+\boldsymbol{u})/(1+m)$, we have

$$\frac{1}{1+m}\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)=\frac{\partial}{\partial\boldsymbol{v}}. \tag{34}$$

It is seen from Eq. (34) that the generator $\hat{\boldsymbol{\sigma}}$ depends only on relative velocity $\boldsymbol{v}$ and that its form is very simple,

$$\hat{\boldsymbol{\sigma}}=-\boldsymbol{v}\times\frac{\partial}{\partial\boldsymbol{v}}=\frac{\partial}{\partial\boldsymbol{v}}\times\boldsymbol{v}. \tag{35}$$

The operator (35), accurate up to constant, is the well-known quantum mechanical operator of angular momentum with commutative relations,

$$[\hat{\sigma}_i,\hat{\sigma}_k]=e_{ikl}\hat{\sigma}_l,\quad [\hat{\sigma}_i,\hat{\sigma}^2]=0,$$
$$[\hat{\sigma}_i,v_k]=e_{ikl}v_l,\quad \left[\hat{\sigma}_i,\frac{\partial}{\partial v_k}\right]=e_{ikl}\frac{\partial}{\partial v_l}, \tag{36}$$
$$\hat{\sigma}_iv_i=v_i\hat{\sigma}_i=0.$$

Mapping $\hat{c}\to\hat{\sigma}$ of Eq. (29) obviously preserves commutative relations,

$$[\hat{\sigma}(\hat{c}_1),\hat{\sigma}(\hat{c}_2)]=\hat{\sigma}[\hat{c}_1,\hat{c}_2]=\hat{\sigma}(\hat{c}(\boldsymbol{n}_1\times\boldsymbol{n}_2)). \tag{37}$$

With the help of expression (30) for the generator, we can obtain the exponential representation of scattering transformation $\hat{T}_s$ in the Hilbert space of functions on bivectors $\xi=(\boldsymbol{\nu},\boldsymbol{u})^T$:

$$\hat{T}_sF(\xi)=F(\hat{S}^{-1}\xi)=e^{\phi\hat{\sigma}}F(\xi). \tag{38}$$

From Eq. (38) it can be easily seen that the equation for $e^{\phi\hat{\sigma}}$ acting on the product of two functions is

$$e^{\phi\hat{\sigma}}F_1(\xi)F_2(\xi)=F_1(\hat{S}^{-1}\xi)F_2(\hat{S}^{-1}\xi j)$$
$$=[e^{\phi\hat{\sigma}}F_1(\xi)][e^{\phi\hat{\sigma}}F_2(\xi)]. \tag{39}$$

Equation (38) allows us to rewrite the collision operator in a new form in which the group structure of a process of particle scattering is explicitly used. To do this we need expressions for invariant measure $d\hat{R}$ and the cosine of the scattering angle μ through parameters ϕ and $\boldsymbol{n}$ (or ϕ and θ, the latter being the angle between vectors $\boldsymbol{v}$ and $\boldsymbol{n}$).

The invariant measure for the group of proper rotations is as follows when rotations are parametrized by the angle of rotation ϕ and the direction $\boldsymbol{n}$ of the rotation axis is [6,7]

$$d\hat{R}=2(1-\cos\phi)d\phi\, d\Omega_n, \tag{40}$$

where

$$d\Omega_n=\sin\theta\, d\theta\, d\varphi\quad (0\leqslant\phi,\theta\leqslant\pi,\ \ 0\leqslant\varphi\leqslant 2\pi).$$

Using Eq. (19), the equation for μ is as follows:

$$\mu=\frac{\boldsymbol{v}\cdot\hat{R}\boldsymbol{v}}{v^2}=\frac{(1-\cos\phi)\boldsymbol{v}\cdot\hat{n}^2\boldsymbol{v}+(\sin\phi)\boldsymbol{v}\cdot\hat{n}\boldsymbol{v}+v^2}{v^2},$$
$$=1-\frac{(1-\cos\phi)[\boldsymbol{n}\times\boldsymbol{v}]^2}{v^2}=1-(1-\cos\phi)\sin^2\theta. \tag{41}$$

Substituting Eq. (38) into Eq. (11) and keeping Eqs. (40) and (41) in mind, we obtain the final form for the Boltzmann collision integral:

$$I(f,\psi)=\int b(v,\mu)[f(\boldsymbol{v}')\psi(\boldsymbol{u}')-f(\boldsymbol{v})\psi(\boldsymbol{u})]\frac{d\hat{R}}{2\pi}d\boldsymbol{u},$$

$$=\int d\boldsymbol{u}\frac{d\hat{R}}{2\pi}b(v,\mu)[e^{-\phi\hat{\sigma}}-1]f(\boldsymbol{v})\psi(\boldsymbol{u})$$

$$=\int d\boldsymbol{u}\frac{d\hat{R}}{2\pi}[e^{-\phi\hat{\sigma}}-1]b(v,\mu)f(\boldsymbol{v})\psi(\boldsymbol{u}). \tag{42}$$

The form of the collision integral given by Eq. (42) provides new opportunities for its consideration. A rather simple generator $\hat{\sigma}$ of the scattering group determines all general properties of the Boltzmann collision operator. We will not further discuss this question here, but only point out that the product of Maxwellian velocity distributions is an eigenfunction of operator $\hat{\sigma}$ with zero eigenvalue,

$$\hat{\sigma}e^{-m_1 v^2/2kT-m_2u^2/2kT}$$

$$=-\frac{1}{1+m}\left\{\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{u})\times\left(\frac{\partial'}{\partial\boldsymbol{v}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)\right\}$$

$$\times e^{-m_1v^2/2kT-m_2u^2/2kT}=0, \tag{43}$$

and that $\hat{\sigma}$ commutates with the three invariants:

$$[\hat{\sigma},v^2]=0,\quad [\hat{\sigma},\mu]=0,\quad [\hat{\sigma},(m\boldsymbol{v}+\boldsymbol{u})]=0. \tag{44}$$

Note that $\hat{\sigma}$ can be written in the following equivalent forms:

$$\hat{\sigma}=\boldsymbol{n}\cdot\left[\frac{\partial}{\partial\boldsymbol{v}}\times\boldsymbol{v}\right]=-\boldsymbol{n}\cdot\left[\boldsymbol{v}\times\frac{\partial}{\partial\boldsymbol{v}}\right]=-[\boldsymbol{n}\times\boldsymbol{v}]\cdot\frac{\partial}{\partial\boldsymbol{v}}$$

$$=-\frac{\partial}{\partial\boldsymbol{v}}\cdot[\boldsymbol{n}\times\boldsymbol{v}]. \tag{45}$$

Also, from Eqs. (36) and (31) we have

$$[\hat{\sigma},\boldsymbol{n}\cdot\boldsymbol{v}]=0, \tag{46}$$

$$[\hat{\sigma},\boldsymbol{v}]=-\boldsymbol{n}\times\boldsymbol{v},\quad [\hat{\sigma},\boldsymbol{n}\times\boldsymbol{v}]=-\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{v}. \tag{47}$$

III. RENORMALIZATION OF THE BOLTZMANN COLLISION OPERATOR

In this section, we present an exact method of rewriting the Boltzmann collision integral in the divergence form based on Eq. (42). Within the framework of this method, we show how to reduce or extract the singularities connected with infinite total collision cross sections related to small-angle scatterings. We consider the Coulomb collisions in detail.

To separate the singular part of the collision integral from the regular one, we do not use the usual Taylor series but rather the Taylor series with a residual term. For any function $\varphi(\alpha)$ with derivatives we can write

$$\varphi(\alpha)=\varphi(0)+\frac{\varphi'(0)}{1!}\alpha+\cdots+\frac{\varphi^{(n-1)}(0)}{(n-1)!}\alpha^{n-1}+\hat{O}_n\varphi(\alpha), \tag{48}$$

where

$$\hat{O}_n\varphi(\alpha)=\left(\int_0^\alpha d\alpha\right)\left(\frac{\partial}{\partial\alpha}\right)^n\varphi(\alpha). \tag{49}$$

Let us consider the operator $[e^{-\phi\hat{\sigma}}-1]$ in Eq. (42). Multiplying the angle ϕ in the exponent function $e^{-\phi\hat{\sigma}}$ by some coefficient α, making use of Eq. (48), and then inserting $\alpha=1$, we have the following:

$$e^{-\phi\hat{\sigma}}=1-\phi\hat{\sigma}+\frac{1}{2!}\phi^2\hat{\sigma}^2+\cdots+\frac{(-1)^{n-1}}{(n-1)!}\phi^{n-1}\hat{\sigma}^{n-1}$$

$$+(-1)^n\phi^n\hat{\sigma}^n\int_0^1 d\alpha_n\int_0^{\alpha_n}d\alpha_{n-1}\cdots$$

$$\times\int_0^{\alpha_2}d\alpha_1 e^{-\alpha_1\phi\hat{\sigma}}. \tag{50}$$

Changing the order of the n-fold integration over α_1, α_2,..., and α_n in Eq. (50) yields

$$\int_0^1 d\alpha_n\int_0^{\alpha_n}d\alpha_{n-1}\cdots\int_0^{\alpha_2}d\alpha_1 e^{-\alpha_1\phi\hat{\sigma}}$$

$$=\int_0^1 d\alpha_1\int_{\alpha_1}^1 d\alpha_2\cdots\int_{\alpha_{n-1}}^1 d\alpha_n e^{-\alpha_1\phi\hat{\sigma}}. \tag{51}$$

Keeping in mind that

$$\int_{\alpha_1}^1 d\alpha_2\cdots\int_{\alpha_{n-1}}^1 d\alpha_n=\frac{(1-\alpha_1)^{n-1}}{(n-1)!}, \tag{52}$$

we finally have the Taylor series of the operator $e^{-\phi\hat{\sigma}}$ with the residual term

$$e^{-\phi\hat{\sigma}}=1-\phi\hat{\sigma}+\frac{1}{2!}\phi^2\hat{\sigma}^2+\cdots+\frac{(-1)^{n-1}}{(n-1)!}\phi^{n-1}\hat{\sigma}^{n-1}$$

$$+\frac{(-1)^n}{n!}\phi^n\hat{\sigma}^n\int_0^1 d\alpha\, q_n(\alpha)e^{-\alpha\phi\hat{\sigma}}, \tag{53}$$

where $n=1,2,...$ and

$$q_n(\alpha)=n(1-\alpha)^{n-1},\qquad \int_0^1 d\alpha\, q_n(\alpha)=1. \tag{54}$$

The measure $b(v,\mu)d\hat{R}$ in the collision integral of Eq. (42) is invariant under the replacement $\hat{R}\to\hat{R}^{-1}$. Therefore, we can symmetrize the factor $e^{-\phi\hat{\sigma}}$ in the integral,

$$\int d\hat{R}\,b(v,\mu)e^{-\phi\hat{\sigma}}=\frac{1}{2}\int d\hat{R}\,b(v,\mu)(e^{-\phi\hat{\sigma}}+e^{\phi\hat{\sigma}})$$

$$=d\hat{R}\,b(v,\mu)\Bigg\{\sum_{k=0}^{2k<n}\frac{1}{(2k)!}\phi^{2k}\hat{\sigma}^{2k}+\frac{1}{2\times n!}\phi^{n}\hat{\sigma}^{n}\times\int_0^1 d\alpha\,q_n(\alpha)(e^{\alpha\phi\hat{\sigma}}+(-1)^n e^{-\alpha\phi\hat{\sigma}})\Bigg\}. \tag{55}$$

Substituting Eq. (55) into Eq. (42), we obtain the following:

$$I(f,\psi)=\int d\boldsymbol{u}\frac{d\hat{R}}{2\pi}b(v,\mu)\Bigg\{\sum_{k=1}^{2k<n}\frac{1}{(2k)!}\phi^{2k}\hat{\sigma}^{2k}+\frac{1}{2\times n!}\phi^{n}\hat{\sigma}^{n}\int_0^1 d\alpha\,q_n(\alpha)[e^{-\alpha\phi\hat{\sigma}}+(-1)^n e^{-\alpha\phi\hat{\sigma}}]\Bigg\}f(\boldsymbol{\nu})\psi(\boldsymbol{u}). \tag{56}$$

Recalling Eq. (31), we can separate quantities that depend on collision parameter $\boldsymbol{n}$ in $\hat{\sigma}^n$ (the same letter n for the order of the residual term and the unit vector $\boldsymbol{n}$ may not cause confusion):

$$\hat{\sigma}^n=(\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}})^n=n_{i_1}\cdots n_{i_n}\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_n}, \tag{57}$$

$$I(f,\psi)=\int d\boldsymbol{u}\Bigg\{\sum_{k=1}^{2k<n}\frac{1}{(2k)!}\langle\phi^{2k}n_{i_1}\cdots n_{i_{2k}}\rangle\hat{\sigma}_{i_1}\cdots\hat{\sigma}_{i_{2k}}+\frac{1}{2\times n!}\langle\phi^n\hat{\sigma}^n(e^{\alpha\phi\hat{\sigma}}+(-1)^n e^{-\alpha\phi\hat{\sigma}})\rangle\Bigg\}f(\boldsymbol{\nu})\psi(\boldsymbol{u}). \tag{58}$$

Here the brackets $\langle\ \rangle$ denote averaging over the parameters of collision.

$$\langle\cdots\rangle=\int d\alpha\frac{d\phi}{\pi}d\Omega_n q_n(\alpha)(1-\cos\phi)b(v,\mu)[\cdots]. \tag{59}$$

Taking into account that the divergence $(\partial/\partial\boldsymbol{u})(\cdots)$ vanishes after integration over $\boldsymbol{u}$, we can rewrite Eq. (56) in the following divergence form:

$$I(f,\psi)=-\frac{\partial}{\partial\boldsymbol{\nu}}\boldsymbol{J}(f,\psi)\quad(n>0), \tag{60}$$

where the flow in the velocity space is given by

$$\boldsymbol{J}(f,\psi)=-\frac{1}{1+m}\int d\boldsymbol{u}\,\boldsymbol{v}\times\Bigg\langle\boldsymbol{n}\Bigg\{\sum_{k=1}^{2k<n}\frac{1}{(2k)!}\phi^{2k}\hat{\sigma}^{2k-1}+\frac{1}{2\times n!}\phi^n\hat{\sigma}^{n-1}[e^{\alpha\phi\hat{\sigma}}+(-1)^n e^{-\alpha\phi\hat{\sigma}}]\Bigg\}\Bigg\rangle f(\boldsymbol{\nu})\psi(\boldsymbol{u}), \tag{61}$$

or in the index form

$$J_i(f,\psi)=-\frac{1}{1+m}e_{ii_0i_1}\int d\boldsymbol{u}\,v_{i_0}\times\Bigg\{\sum_{k=1}^{2k<n}\frac{1}{(2k)!}\langle\phi^{2k}n_{i_1}\cdots n_{i_{2k}}\rangle\hat{\sigma}_{i_2}\cdots\hat{\sigma}_{i_{2k}}+\frac{1}{2\times n!}\langle\phi^n n_{i_1}\hat{\sigma}^{n-1}(e^{\alpha\phi\hat{\sigma}}+(-1)^n e^{-\alpha\phi\hat{\sigma}})\rangle\Bigg\}f(\boldsymbol{\nu})\psi(\boldsymbol{u}). \tag{62}$$

IV. EXAMPLES

Here we consider the general renormalized expression (56) for the Boltzmann collision integral for the most important cases of $n=0$, 1, 2, and 4, where n is the series termination number.

Case $n=0$.

$$I(f,\psi)=\int d\boldsymbol{u}\frac{d\phi}{\pi}d\Omega_n(1-\cos\phi)b(v,\mu)\times\left(\frac{e^{\phi\hat{\sigma}}+e^{-\phi\hat{\sigma}}}{2}-1\right)f(\boldsymbol{\nu})\psi(\boldsymbol{u}). \tag{63}$$

Equation (63) is the usual form of the Boltzmann collision integral that is expressed through the generator of the scattering group, where the principle of detailed equilibrium is explicitly satisfied.

Case $n=1$. Inserting the value $n=1$ into general Eq. (56), we have a renormalized equation for the Boltzmann collision integral with respect to the flow in the velocity space:

$$I(f,\psi)=-\frac{\partial}{\partial\boldsymbol{\nu}}\boldsymbol{J}(f,\psi), \tag{64}$$

where

$$\boldsymbol{J}(f,\psi)=\frac{1}{1+m}\int d\boldsymbol{u}\left\langle\boldsymbol{n}\times\boldsymbol{v}\left\{\frac{1}{2}\phi(e^{\alpha\phi\hat{\sigma}}-e^{-\alpha\phi\hat{\sigma}})\right\}\right\rangle f(\boldsymbol{\nu})\psi(\boldsymbol{u}) \tag{65}$$

$$=\frac{1}{1+m}\int d\boldsymbol{u}\frac{d\phi}{\pi}d\Omega_n d\alpha(1-\cos\phi)b(v,\mu)\times[\boldsymbol{n}\times\boldsymbol{v}]\frac{\phi}{2}(e^{\alpha\phi\hat{\sigma}}-e^{-\alpha\phi\hat{\sigma}})f(\boldsymbol{\nu})\psi(\boldsymbol{u}) \tag{66}$$

$$=-\frac{1}{1+m}\int d\boldsymbol{u}\frac{d\phi}{\pi}d\Omega_n d\alpha(1-\cos\phi)b(v,\mu)\times[\boldsymbol{n}\times\boldsymbol{v}]\frac{\phi}{2}[f(\boldsymbol{\nu}'_{+\alpha})\psi(\boldsymbol{u}'_{+\alpha})-f(\boldsymbol{\nu}'_{-\alpha})\psi(\boldsymbol{u}'_{-\alpha})]. \tag{67}$$

The Boltzmann equation can be rewritten in the form of the Liouville equation with the help of the divergence form of the collision integral (60):

$$\frac{\partial f}{\partial t}+\boldsymbol{\nu}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+\frac{\partial}{\partial \boldsymbol{\nu}}\cdot\left(\frac{e_1}{m_1}\boldsymbol{E}+\frac{e_1}{m_1 c}\boldsymbol{\nu}\times\boldsymbol{B}+\boldsymbol{g}+\frac{1}{m_1}\boldsymbol{F}_{\rm coll}\right)f=0. \tag{68}$$

Note that in addition to the usual electromagnetic force and gravity force, we have nonlocal friction force $\boldsymbol{F}_{\rm coll}$, which depends on the distribution functions $f(\boldsymbol{\nu})$ and $\psi(\boldsymbol{u})$. In the case of $n=1$ this force has the following expression:

$$\boldsymbol{F}_{\rm coll}=\frac{\boldsymbol{J}}{m_1 f(\boldsymbol{\nu})}=-\frac{1}{m_1+m_2}\int d\boldsymbol{u}\frac{d\phi}{\pi}d\Omega_n d\alpha(1-\cos\phi)b(v,\mu)\times[\boldsymbol{n}\times\boldsymbol{v}]\frac{\phi}{2}\frac{[f(\boldsymbol{\nu}'_{+\alpha})\psi(\boldsymbol{u}'_{+\alpha})-f(\boldsymbol{\nu}'_{-\alpha})\psi(\boldsymbol{u}'_{-\alpha})]}{f(\boldsymbol{\nu})}, \tag{69}$$

where the postcollisional velocities can be expressed as

$$\boldsymbol{\nu}'_{\pm\alpha}=\boldsymbol{\nu}+\frac{[(1-\cos\alpha\phi)\hat{n}^2\pm(\sin\alpha\phi)\hat{n}](\boldsymbol{\nu}-\boldsymbol{u})}{1+m}. \tag{70}$$

$$\boldsymbol{u}'_{\pm\alpha}=\boldsymbol{u}-\frac{m[(1-\cos\alpha\phi)\hat{n}^2\pm(\sin\alpha\phi)\hat{n}](\boldsymbol{\nu}-\boldsymbol{u})}{1+m}. \tag{71}$$

Recall that

$$\hat{n}\boldsymbol{\nu}\equiv\boldsymbol{n}\times\boldsymbol{\nu},$$

$$\hat{R}=e^{\phi\hat{n}}=(1-\cos\phi)\hat{n}^2+(\sin\phi)\hat{n}+1.$$

For the Coulomb collisions, the collision indicatrix takes the following form using the Rutherford formula

$$b(v,\mu)=(1+m)^2\left(\frac{e_1e_2}{m_1}\right)^2\frac{1}{v^3(1-\mu)^2}=(1+m)^2\left(\frac{e_1e_2}{m_1}\right)^2\times\frac{1}{v^3(1-\cos\phi)^2(1-\cos\theta)^2(1+\cos\theta)^2}. \tag{72}$$

The probability of collision is given by the measure

$$\frac{d\hat{R}}{2\pi}b(v,\mu)=(1+m)^2\left(\frac{e_1e_2}{m_1}\right)^2\times\frac{d\phi\, d\Omega_n}{\pi v^3(1-\cos\phi)(1-\cos\theta)^2(1+\cos\theta)^2}, \tag{73}$$

for $n=1$ we have $q_1(\alpha)=1$, and the flow $\boldsymbol{J}$ takes the form

$$\boldsymbol{J}(f,\psi)=-(1+m)^2\left(\frac{e_1e_2}{m_1}\right)^2\int d\boldsymbol{u}\times\frac{d\phi\, d\Omega_n d\alpha}{\pi v^3(1-\cos\phi)(1-\cos\theta)^2(1+\cos\theta)^2}\times\frac{\phi}{2}\frac{[\boldsymbol{n}\times\boldsymbol{v}]}{(1+m)}[f(\boldsymbol{\nu}'_{+\alpha})\psi(\boldsymbol{u}'_{+\alpha})-f(\boldsymbol{\nu}'_{-\alpha})\psi(\boldsymbol{u}'_{-\alpha})]. \tag{74}$$

Now we have an essentially new form of the collision integral. The collision integral is represented by the divergence of the flow in the velocity space. A new parameter, $0\leqslant\alpha\leqslant 1$, that reduces the strength of collision is introduced. The obtained equation may be convenient for Monte Carlo simulations.

Case n=2. The flow in the velocity space is given by the following equation:

$$\boldsymbol{J}(f,\psi)=\frac{1}{1+m}\int d\boldsymbol{u}\left\langle\frac{1}{4}\phi^2(\boldsymbol{n}\times\boldsymbol{v})\hat{\sigma}(e^{\alpha\phi\hat{\sigma}}+e^{-\alpha\phi\hat{\sigma}})\right\rangle\times f(\boldsymbol{\nu})\psi(\boldsymbol{u}) \tag{75}$$

$$=-\frac{1}{(1+m)^2}\int d\boldsymbol{u}\left\langle\frac{1}{4}\phi^2(\boldsymbol{n}\times\boldsymbol{v})(\boldsymbol{n}\times\boldsymbol{v})\cdot\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)[f(\boldsymbol{\nu}'_{+\alpha})\psi(\boldsymbol{u}'_{+\alpha})+f(\boldsymbol{\nu}'_{-\alpha})\psi(\boldsymbol{u}'_{-\alpha})]\right\rangle. \tag{76}$$

For $n=2$, the probability function $q_n(\alpha)$ for the reducing parameter α is $q_1(\alpha)=1-\alpha$ from Eq. (54). If we change the probability function $q_1(\alpha)$ into the δ function $\delta(\alpha)$, which is far from reality, then we have the collision integral in the Landau-Fokker-Planck form,

$$\boldsymbol{J}(f,\psi)=-\frac{1}{2(1+m)^2}\int d\boldsymbol{u}\langle\phi^2(\boldsymbol{n}\times\boldsymbol{v})(\boldsymbol{n}\times\boldsymbol{v})\rangle\cdot\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)f(\boldsymbol{\nu})\psi(\boldsymbol{u}). \tag{77}$$

Making use of commutation rule $[(\boldsymbol{n}\times\boldsymbol{v}),\hat{\sigma}]=\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{v}$, and recalling that $\hat{\sigma}$ can be expressed as

$$\hat{\sigma}=-\frac{1}{1+m}\left(\frac{\partial}{\partial\boldsymbol{\nu}}-m\frac{\partial}{\partial\boldsymbol{u}}\right)\cdot(\boldsymbol{n}\times\boldsymbol{v}), \tag{78}$$

we can rewrite Eq. (77) in the half-divergence form

$$\boldsymbol{J}(f,\psi)=\frac{1}{(1+m)}\int d\boldsymbol{u}\left\langle\frac{1}{4}\phi^2\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{v}[f(\boldsymbol{\nu}'_{+\alpha})\psi(\boldsymbol{u}'_{+\alpha})+f(\boldsymbol{\nu}'_{-\alpha})\psi(\boldsymbol{u}'_{-\alpha})]\right\rangle-\frac{1}{(1+m)^2}\frac{\partial}{\partial\boldsymbol{\nu}}\cdot\int d\boldsymbol{u}\times\left\langle\frac{1}{4}\phi^2(\boldsymbol{n}\times\boldsymbol{v})(\boldsymbol{n}\times\boldsymbol{v})[f(\boldsymbol{\nu}'_{+\alpha})\psi(\boldsymbol{u}'_{+\alpha})\right.$$

$$+f(\boldsymbol{\nu}'_{-\alpha})\psi(\boldsymbol{u}'_{-\alpha})]\bigg\rangle. \tag{79}$$

Case n=4. The collision integral becomes

$$I(f,\psi)=\int d\boldsymbol{u}\bigg\{\frac{1}{2!}\langle\phi^2 n_{i_1}n_{i_2}\rangle\hat{\sigma}_{i_1}\hat{\sigma}_{i_2}+\frac{1}{2\times 4!}\langle\phi^4\hat{\sigma}^4(e^{\alpha\phi\hat{\sigma}}+e^{-\alpha\phi\hat{\sigma}})\rangle\bigg\}f(\boldsymbol{\nu})\psi(\boldsymbol{u}), \tag{80}$$

and the flow in velocity space is

$$J_i(f,\psi)=-\frac{1}{1+m}e_{ii_0i_1}\int d\boldsymbol{u}\,v_{i_0}\bigg\{\frac{1}{2!}\langle\phi^2 n_{i_1}n_{i_2}\rangle\hat{\sigma}_{i_2}+\frac{1}{2\times 4!}\langle\phi^4 n_{i_1}\hat{\sigma}^3(e^{\alpha\phi\hat{\sigma}}+e^{-\alpha\phi\hat{\sigma}})\rangle\bigg\}f(\boldsymbol{\nu})\psi(\boldsymbol{u}). \tag{81}$$

Let us evaluate the average $\langle\phi^2 n_{i_1}n_{i_2}\rangle$ in Eq. (81). A general form of the symmetric tensor, which depends only on one vector, is

$$\langle\phi^2 n_i n_k\rangle=A\,\delta_{ik}+B\,\frac{v_i v_k}{v^2}. \tag{82}$$

To determine the unknown coefficients A and B, we have two conditions,

$$\delta_{ik}\langle\phi^2 n_i n_k\rangle=\langle\phi^2\rangle,\quad \frac{v_i v_k}{v^2}\langle\phi^2 n_i n_k\rangle=\langle\phi^2\cos^2\theta\rangle, \tag{83}$$

which give us two simple equations for the coefficients in Eq. (82), i.e.,

$$3A+B=\langle\phi^2\rangle\quad\text{and}\quad A+B=\langle\phi^2\cos^2\theta\rangle. \tag{84}$$

The final result of averaging is as follows:

$$\langle\phi^2 n_i n_k\rangle=\frac{1}{2}\langle\phi^2\sin^2\theta\rangle\delta_{ik}+\frac{1}{2}\langle\phi^2(3\cos^2\theta-1)\rangle\frac{v_i v_k}{v^2}. \tag{85}$$

Recalling that $v_i\hat{\sigma}_i=\hat{\sigma}_i v_i=0$, we have

$$\langle\phi^2 n_i n_k\rangle\hat{\sigma}_k=\tfrac{1}{2}\langle\phi^2\sin^2\theta\rangle\hat{\sigma}_i, \tag{86}$$

$$\begin{aligned}\langle\phi^2 n_i n_k\rangle\hat{\sigma}_i\hat{\sigma}_k&=\tfrac{1}{2}\langle\phi^2\sin^2\theta\rangle\hat{\sigma}_i\hat{\sigma}_i\\&=\frac{\langle\phi^2\sin^2\theta\rangle}{2(1+m)^2}\left(\frac{\partial}{\partial\nu_i}-m\frac{\partial}{\partial u_i}\right)\\&\quad\times(v^2\delta_{ik}-v_i v_k)\left(\frac{\partial}{\partial\nu_k}-m\frac{\partial}{\partial u_k}\right).\end{aligned} \tag{87}$$

Now the Boltzmann collision integral for the case of $n=4$ is represented by the sum of the two terms:

$$\begin{aligned}I(f,\psi)=&\frac{1}{2(1+m)^2}\frac{\partial}{\partial\nu_i}\int d\boldsymbol{u}\frac{\langle\phi^2\sin^2\theta\rangle}{2}(v^2\delta_{ik}-v_i v_k)\\&\times\left(\frac{\partial}{\partial\nu_k}-m\frac{\partial}{\partial u_k}\right)f(\boldsymbol{\nu})\psi(\boldsymbol{u})\\&+\frac{1}{2\times 4!}\int d\boldsymbol{u}\langle\phi^4\cdot\hat{\boldsymbol{\sigma}}^4(e^{\alpha\phi\hat{\sigma}}+e^{-\alpha\phi\hat{\sigma}})\rangle f(\boldsymbol{\nu})\psi(\boldsymbol{u}).\end{aligned} \tag{88}$$

Using Eq. (73), we have

$$\begin{aligned}\frac{1}{2}\langle\phi^2\sin^2\theta\rangle=&(1+m)^2\left(\frac{e_1e_2}{m_1}\right)^2\\&\times\int\frac{\phi^2 d\phi\, d\cos\theta}{v^3(1-\cos\phi)(1-\cos\theta)(1+\cos\theta)}\end{aligned} \tag{89}$$

The integral in Eq. (89) diverges logarithmically as $\theta\rightarrow 0$ or π. To avoid this, we introduce the cutoff angle $\theta_{\min}$ and consider the range $\theta_{\min}<\theta<\pi-\theta_{\min}$, which is usual for Coulomb collisions. Due to the transversability condition $v_i\hat{\sigma}_i=0$, we can change the vector $\boldsymbol{n}$ into the vector $\boldsymbol{n}_\perp=\boldsymbol{n}-\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n}/v^2)$ in the operator $\hat{\sigma}=\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}$. A squared norm of this vector is $\boldsymbol{n}_\perp^2=1-\cos^2\theta$ and $\hat{\sigma}$ commutates with $\cos\theta$. It implies that

$$\hat{\sigma}^4=(\boldsymbol{n}_\perp\cdot\hat{\boldsymbol{\sigma}})^4=\sin^4\theta\left(\frac{\boldsymbol{n}_\perp}{|\boldsymbol{n}_\perp|}\cdot\hat{\boldsymbol{\sigma}}\right)^4. \tag{90}$$

Different from the first term of Eq. (88), the second term of Eq. (88) does not diverge as $\theta\rightarrow 0$ or π. The first term in Eq. (88) is in essence the Landau-Fokker-Plank collision integral [13,14]. The only difference is that in the present formulation we have the new effective collision frequency $\langle\phi^2\sin^2\theta\rangle/2$ instead of the transport collision frequency $\langle 1-\mu\rangle=\langle(1-\cos\phi)\sin^2\theta\rangle$ introduced in Refs. [13] and [14]. At small ϕ, which always means small scattering cosine μ but not vice versa, the two frequencies coincide. For the present effective collision frequency we can easily prove the inequality

$$\langle 1-\mu\rangle<\frac{\langle\phi^2\sin^2\theta\rangle}{2}<\frac{\pi^2}{4}\langle 1-\mu\rangle. \tag{91}$$

The small difference between the present and previous [13,14] frequencies is not very important for practical physical problems, but it is important that the lack of coincidence of these two frequencies proves that the Landau series

[13,14] is divergent.

In conclusion, it is worth emphasizing that the renormalized equations for the Boltzmann collision integral obtained in the present paper are not approximate expansions but exact ones that are especially useful when a total collision cross section diverges and the term *collision* is badly determined.

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